




Define  $\delta_i=1$  if  $t_i \leq c_i$  (uncensored)  
 $\delta_i=0$  if  $t_i > c_i$  (censored)


Then likelihood  $L(\lambda) = \prod_{i=1}^n [\lambda e^{-\lambda t_i}]^{\delta_i} [e^{-\lambda c_i}]^{1-\delta_i}$

$$\Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n \{\delta_i t_i + (1 - \delta_i) c_i\}}$$


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$$\hat{\lambda} = \frac{\text{total number of deaths observed}}{\text{total time alive of all patients in the study}}$$


holds for censored and uncensored cases



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- Distribution of  $\hat{\lambda}$  not easy
  - (not based on simple sum of exponentials)
  - Use general theory of MLE for approximate formula :

$$\text{var}(\hat{\lambda}) \approx \frac{\hat{\lambda}^2}{\sum_{i=1}^n (1 - e^{-\hat{\lambda} c_i})}$$


$$\text{var}(\hat{\lambda}) \approx \frac{\hat{\lambda}^2}{\sum_{i=1}^n \delta_i} \quad \text{numerically same [almost]}$$


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- $100(1 - \alpha)\%$  Confidence Interval for  $\lambda$  is

$$\hat{\lambda} \pm z_{1-\alpha/2} \times \text{s.e.}(\hat{\lambda})$$

Where  $\text{s.e.}(\hat{\lambda}) = \sqrt{\text{var}(\hat{\lambda})}$




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- Note use of formula

$$\text{var}\{g(\hat{\lambda})\} \approx \left[ \frac{d}{d\lambda} g(\lambda) \right]_{\lambda=\hat{\lambda}}^2 \text{var}(\hat{\lambda})$$

for finding CIs of functions  $g(\lambda)$  of  $\lambda$   
 e.g.  $\mu = \lambda^{-1} = E[T]$  the mean lifetime  
 $g(\lambda) = \lambda^{-1}$  so  $g'(\lambda) = -\lambda^{-2} = -\mu^2$  so


$$\text{var}(\hat{\mu}) \approx \frac{\hat{\mu}^2}{\sum_{i=1}^n (1 - e^{-c_i/\hat{\mu}})} \quad \text{or} \quad \text{var}(\hat{\mu}) \approx \frac{\hat{\mu}^2}{\sum_{i=1}^n \delta_i}$$


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- Example

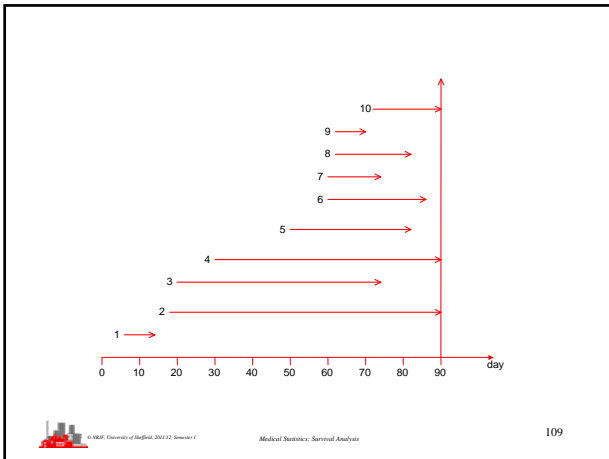
- ♦ 10 patients, 3 censored (2<sup>nd</sup>, 4<sup>th</sup> & 10<sup>th</sup>)

Patient no.	1	2	3	4	5	6	7	8	9	10
Entry time	9	18	20	30	49	59	59	60	61	69
Survival time $t_i$	2	.	51	.	33	27	14	24	4	.
max possible $c_i$	81	72	70	60	41	31	31	30	29	21
$\delta_i$	1	0	1	0	1	1	1	1	1	0



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- Thus  $\sum \delta_i = 7$  deaths on study.  
 $\sum \delta_i t_i = 155$ ,  $\sum (1 - \delta_i) c_i = 153$   
 $\hat{\lambda} = \frac{7}{308} = 0.023$  per day  
 $\hat{\mu} = \frac{308}{7} = 44.0$  days  
 confidence intervals etc from formula

- data  $\{(t_i, \delta_i); i=1, \dots, n\}$ 
  - $\delta_i = 1$  if death occurred, 0 if censored
$$L(\theta) = \prod_{\text{deaths}} f(t_i) \prod_{\text{censored}} S(t_i)$$

$$= \prod_{\text{deaths}} h(t_i) S(t_i) \prod_{\text{censored}} S(t_i)$$

$$= \prod_{i=1}^n [h(t_i)]^{\delta_i} S(t_i)$$

- This general formula holds for all distributions (lognormal, Weibull, gamma, Gumbel,.....)
- All require numerical estimation
  - Minitab, SPSS, S-PLUS

- Summary**
  - Life tables
    - Grouped data
    - Allow for censoring by adjusting # at risk
  - Kaplan-Meier
    - Individual data
    - Uncensored case = 1 - empirical CDF
    - Express as a product of terms  $[1 - d_i/r_i]$
    - Censored case by adjusting # at risk

- Kaplan-Meier estimates are the key exploratory tool for censored data
  - Cannot draw histograms of censored data
    - Don't know which bin to put a censored in
- Kaplan-Meier Plot is always step 1 in analysing censored data



- Parametric models
  - ◆ Estimate parameters by MLE
  - ◆ Uncensored observations contribute  $f(t_i)$  &
  - ◆ Censored contribute  $S(t_i)$  to likelihood
  - ◆ Use MLE theory for standard errors
  - ◆ Plug in MLEs for other functions of  $\theta$
  - ◆ Use formula for s.e.[ $g(\theta)$ ]

